CHILDREN'S APPROACHES TO MATHEMATICAL PROBLEM SOLVING

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This paper reports on one aspect of a study designed to explore the role of metacognition in fourth grade children's mathematical problem solving. A response mapping technique was developed to represent individual problem solving episodes in terms of Flavell's (1981) model of cognitive monitoring. The consistency of the patterns observed in the individual models suggested a relationship between conceptual and procedural knowledge of a cognitive and metacognitive kind. A meta-model was proposed to describe and explain the nature of this inferred relationship. The paper provides an outline of the response mapping technique, a summary of the meta-model, and an example of the response map data. The analysis supported the view that children's approaches to mathematical problem solving can be described in terms of the four generic approaches identified by the meta-model, that is, a Solver's approach (high conceptual-high procedural), a Diver's approach (high conceptual-low procedural), a Player's approach (low conceptual-high procedural), and a Survivor's approach (low conceptual-low procedural).

BACKGROUND

The work reported here derives from two teaching experiments involving upper primary grade children and their teachers. The first, conducted from an information-processing perspective, was designed to investigate the effects of metacognitive training on children's mathematical problem solving performance over a ten week period. Two questions prompted the first study, (1) to what extent was metacognition a "driving force" in children's mathematical problem solving, and (2) to what extent could children's cognitive awareness and ability to monitor and regulate their actions be improved through training. A ten-week teaching experiment involving three grade four classes and one grade six class was designed to address these questions (Siemon, 1986). While the results of the initial study supported the view that metacognition as it was defined, is a "force" governing mathematical problem solving behaviour, and that for some students enhanced metacognition can be achieved by training, the complexity of the problem solving behaviour observed suggested that metacognition is not a single or even bi-valued "driving force", but a multiplicity of complex "forces"and relationships not all of which may be operating in the same direction at the same time. The second, and more major of the two studies was conducted from a constructivist's perspective. A year long teaching experiment was designed to explore the role of metacognition in children's mathematical problem solving, specifically, the sufficiency and viability of a theoretical meta-model of children's approaches to mathematical problem solving derived from a *post hoc* analysis of the individual interview data obtained from the first study. A summary of the meta-model and the response mapping technique which helped generate it will be addressed in this paper.

THE FIRST STUDY

The uniqueness and significance of children's existing knowledge, beliefs, goals, and motivations, in relation to their subsequent learning and problem solving attempts was powerfully demonstrated by the first study. As a result, problem solving was seen as an exercise in negotiating meaning, not as a "process" to be taught and exercised in schools because it was the "focus" of the decade. It was apparent that a theory of learning as opposed to a theory of instruction was needed, and a view of metacognition which recognised the importance of existing metacognitive knowledge, beliefs and experience in the construction of mathematical meanings and solution strategies. The underlying theory of learning was found in *constructivism* (for example, Cobb, 1987). The more encompassing view of metacognition was found in Flavell's (1981) Model of Cognitive Monitoring, in which he includes the two recognised aspects of metacognition, *metacognitive knowledge* and *metacognitive experience*, and adds two other components, *cognitive goals* and *cognitive actions*. From a constructivist's perspective of problem solving, this model seemed to provide a better explanation of what had been observed in the protocols than the two component models so prevalent in the problem solving literature conducted from an information processing perspective (for example, Schoenfeld, 1983; and Garofalo and Lester, 1985).

THE RESPONSE MAPPING TECHNIQUE

On the basis of this re-evaluation of research goals and meanings, a protocol analysis scheme was developed which recognised the interactive role of metacognitive knowledge, metacognitive experiences, cognitive goals and cognitive actions. The scheme was adapted from the diagrams used by Biggs and Collis (1982) to characterise the various levels in the SOLO Taxonomy. While the basic system of recording used to generate the maps is similar to that used by Collis and Watson (1989), there are some important differences. For instance, the response mapping technique does not assume that information or procedures not given in the problem stem but essential for solution, are available to the solver if they have been taught or can be deemed to be in the "universe of discourse" (Collis and Watson, 1989, p.182). The technique is not concerned with identifying the particular SOLO level of a child's response nor with classifying problems in terms of SOLO levels on the basis of written test data. It was developed to accommodate clinical interview data obtained from 9 to 10 year olds. A major difference in the technique, at least in its formative stages, is the inclusion of a summarised record of interview which can be used to support an inference or to elaborate a response. The following symbols are used to construct the response maps developed for this study: **l** - information provided in the problem stem or by the researcher, which may or may not be relevant to solution, and which may or may not be attended to by the solver; and **o** - information observed or inferred on the basis of transcript evidence or the child's actions in the course of a problem solving attempt, that is, information provided or generated by the solver which may or may not be relevant to solution.

In this context, *information* includes: numerical data, problem conditions, questions, cognitive goals, items of cognitive or metacognitive knowledge (including beliefs, concepts, skills, processes and strategies), and the products of cognitive actions and metacognitive experiences. Two forms of information were proposed: *presented* information and *generated* information.

Two types of presented information were identified, These are listed on the left hand side of the response map. The first type is information provided in the problem stem or by the researcher (indicated by the symbol **l**). The second type is information provided by the solver in the form of an inferred *cognitive goal* or item of *cognitive* or *metacognitive knowledge* (indicated by the symbol **o**), for example, the decision to represent a problem by drawing a diagram, or the belief that because it is a mathematics problem, an algorithm is required.

Generated information is information provided by the solver in the course of the problem solving enterprise as a result of a *cognitive action* or *metacognitive experience*, for example, a drawing, an answer to the first step of a multiple step problem, or suddenly recognising that a cognitive goal or action is inappropriate. Items of generated information are also represented by the symbol **o**, but they appear in various locations to the right of the list.

Classifying cognitive goals and items of metacognitive knowledge as items of presented information recognises that the solver's knowledge, beliefs and prior experience play a powerful role in interpreting and utilising the result of any cognitive action or metacognitive experience. That this distinction is worth making is strongly supported by the transcript data from the first study which suggests that differences in children's approaches to problem solving are related to differences in the ways and extent to which cognitive goals and cognitive actions are generated, retrieved and monitored.

The maps are constructed from the transcripts of interviews and any written record that the student might have made during the problem solving episode. In general, a separate map is constructed for each problem attempted. Where necessary, an abbreviated transcript is presented with the map. "Expert" maps may be generated for the purposes of discussion or elaboration, but no assumptions are made about the uniqueness or "appropriateness" of such maps. An example of an expert map for the *Incy Wincy p*roblem, more widely recognised as the frog in the well problem, appears in Figure 1. The items of presented information are listed on the left hand side of the map. They include the data and question from the problem as well as the items presumed to be brought to the situation by the expert. That is, the recognition (metacognitive experience) that this problem is similar to the frog in the well problem (metacognitive task knowledge), and that it can be solved by representing the situation in some way and by acting it out (metacognitive strategy x task knowledge).

The decisions, to represent the situation and to act it out, are cognitive goals. The implementations of the decisions are cognitive actions which produce items of generated information, a diagram in the first instance, and a solution in the second. These appear to the right of the list. Items observed or inferred as being responsible for prompting a particular cognitive action or item of generated information are linked by lines to the generated item(s) concerned.

Incy Wincy Spider:			
14 metre pipe up 5 m/day, down 2 m/night How many days to reach the top?	I I I I		
like F.F. problem represent problem act it out	O O O	O draw a diagram, count, record	O arrives at the top on the fourth day
PRESENTED INFORMATION		***GENERATED***	***INFORMATION***

Figure 1: An 'expert' response map for the *Incy Wincy* problem

As each response map was constructed by the researcher in an attempt to explain the child's observed and/or inferred behaviour, no claim is made for the independence and reliability of this technique. Rather, a claim for generalisability is made on the basis of the number of interviews conducted and analysed in this way and on the basis of the broad patterns that were observed over time and problem type.

A META-MODEL OF CHILDREN'S APPROACHES TO PROBLEM SOLVING

The relatively consistent patterns observed in the individual models of problem solving provided by the response maps suggested there was some value in identifying and comparing individual children's problem solving attempts on the basis of the actual solution attempt made rather than on the basis of some pre-set criteria about the perceived nature of the problem or task itself. For example, observable patterns in Julia's response maps suggested that it was her knowledge and beliefs, both metacognitive and cognitive, that was driving her problem solving efforts more so than the particular demands of the task. Cobb's (1987) observation that our "primary objective is to develop explanations that either account for novel, unanticipated observations or resolve conceptual inconsistencies and contradictions within the theory or model." (p.31), prompted the decision to construct a meta-model that was consistent with a constructivist's perspective of problem solving.

From the *post hoc* analysis of the individual interview data it was apparent that different children appeared to attend to and value qualitatively different aspects of the problem solving experience. Julia, for example, seemed overly concerned with procedures and algorithms (procedural knowledge of a cognitive and metacognitive kind), as opposed to understanding and representing the problem meaningfully (conceptual knowledge of a cognitive and metacognitive kind). Julia's response maps revealed that she was more likely to implement a range of cognitive actions on a trial and error basis than to stop and think about what she was attempting to do or what her actions might mean. While she demonstrated considerable skill in monitoring the implementation of her actions, she did not display anywhere near the same commitment to the formulation and evaluation of cognitive goals in relation to the problem's conditions and/or questions. What appeared to be driving her actions was her beliefs about the nature and purpose of school mathematics (prior metacognitive knowledge), for example, her view that doing mathematics is not concerned with meaning: "we don't do meanings in maths". Other case studies suggested a predisposition to value and attend to the construction of meaning. For example, Annie and Nancy (Siemon, 1991), demonstrated a preparedness to thoroughly analyse problem statements before making any decision about what needed to done. This valuing was reflected in their response maps which indicated that both girls tended to formulate and evaluate cognitive goals which in turn informed their selection and execution of a range of cognitive actions. The apparent role of conceptual and/or procedural knowledge, both of a cognitive and metacognitive kind, in the formation and operation of these predispositions, suggested that Julia's predisposition or approach to problem solving could be described as *low conceptual/high procedural and* Annie's approach as *high conceptual/high procedural.* Further analysis suggested that Annie's earlier approach might be classified as *high conceptual/low procedural* which led to the conjecturing of a fourth possible approach, *low conceptual/low procedural.*

It was postulated that a *high conceptual/high procedural* approach would be typified by a high instance of effective goal setting linked to the implementation of an appropriate range of cognitive actions. Instances of cognitive monitoring, both with respect to goals and actions, would also tend to be associated with this approach. A *high conceptual/low procedural* approach was hypothesised to be typified by a relatively high instance of appropriate goal setting, but a correspondingly low level of implementation of appropriate procedures and actions. Where there were instances of cognitive monitoring they would be more likely to be directed at cognitive goals than at cognitive actions. By contrast, a *low conceptual/high procedural* approach was believed to be typified by infrequent instances of appropriate goal setting, but a correspondingly high level of implementation of appropriate procedures and actions. In this case, where there were instances of cognitive monitoring, they would be more likely to be directed at cognitive actions than at cognitive goals. A *low conceptual/low procedural* approach would be typified by low levels of appropriate goal setting and implementation and few, if any, instances of cognitive monitoring.

A systematic examination of all of the response maps generated from the first study transcripts and records of interview revealed that virtually all of them could be described in terms of either one or other of the four approaches hypothesised. The transcripts were then examined qualitatively with respect to such notions as impulsiveness and reflectivity, field dependence-independence, and approaches to learning based on motivation and type of strategy usage (Biggs and Telfer, 1987; Marton and Saljo, 1976). The meta-model which resulted from this empirical and theoretical analysis is shown in Figure 2. It reflects the major dimensions along which the response maps varied and lists the attributes believed to be characteristic of each approach as a result of the qualitative analysis of the individual interview data with respect to the literature on cognitive style. The suggestion of orthogonality is deliberate as the approaches are not meant to imply discrete, mutually exclusive entities, but *tendencies* towards particular behaviours rather than others, in relation to a specific task at a given point in time. To simplify further discussion, it was decided to name the four approaches as the Solver's Approach *(high conceptual/high procedural)*, the Diver's Approach *(high conceptual/low procedural)*, the Player's Approach *(low conceptual/high procedural)*, and the Survivor's Approach *(low conceptual/low procedural).*

CONCEPTUAL KNOWLEDGE

metacognitive knowledge and cognitive goals

Diver:
High conceptual/
Low Procedural
cognitive goals attended to more than cognitive actions
comprehension strategies: checks, monitors, plans, predicts, links, reflects, introspects,
more on knowledge than on actions
access to a variety of strategies, not always well used
identifies goals
tends to synthesis and analyse data
some tendency to conceptually-driven premature closure
questions directed more at goals than actions
some undirected actions
uses labels, understands structure
deep approach to learning
extended locus of control for actions

Solver:
High Conceptual/
High Procedural
attends to cognitive goals and actions

comprehension and regulation strategies: checks, predicts, monitors, plans, links,
reflects, on knowledge and actions.
uses a variety of strategies knowingly

identifies goals and appropriate actions
synthesises and analyses data
strong tendency to persist until reasonable solution obtained
questions directed at goals and actions
directed manipulation
uses numbers, labels, structure
deep-achieving approach to learning
internal locus of control

Survivor:
Low Conceptual/ Low Procedural
unlikely to attend to either cognitive goals actions cognitive goals
tendency to remember and replicate, but experiences difficulty
strong tendency to premature closure closure
tends not to check, monitor, reflect or predict on actions or goals goals
experiences difficulty identifying goals
experiences difficulty identifying appropriate actions
more likely to synthesise than analyse
little or no questioning

surface approach to learning
external locus of control for knowledge and actions

Player:
Low Conceptual/ High Procedural
cognitive actions attended to more than

remembers and replicates, often quite effectively
tendency to procedurally-driven premature

regulation strategies: checks, monitors, predicts, reflects, more on actions than on

tends to assume goals
tends to try a range of actions

more likely to synthesise than analyse.
questions directed more at actions than at goals
surface-achieving approach to learning
external locus of control for knowledge

PROCEDURAL KNOWLEDGE

metacognitive experiences and cognitive actions

Figure 2: A meta-model of children's approaches to problem solving

THE SUFFICIENCY AND VIABILITY OF THE META-MODEL

The second study was similar to the earlier study in that the teacher conducted at least one problem solving session per week which was video-taped. The program explored a variety of problem types and strategies and provided an accessible model of cognitive monitoring in terms of the ASK-THINK-DO problem solving cycle (see Barry, Booker, Parry and Siemon, 1985) . Key components of the program were (1) the mathematics content of the particular problems considered was selected by the teacher to support her curriculum objectives, (2) the problem solving process was specifically talked about in terms of the ASK-THINK-DO cycle and modelled by the teacher whenever a strategy was reviewed or introduced, and (3) key questions, strategies and observations about problem structure were discussed and recorded on a large ASK-THINK-DO problem solving chart which was on constant display in the classroom. Problems were worked on individually, in small groups or as a class. Twelve children were selected to be interviewed on a regular basis. The interviews consisted of a reflective review of a problem considered in class followed by an attempt at a similar or related problem. Response maps were constructed for all interviews conducted (155 interviews over 15 tasks). The meta-model was judged to be sufficient and viable if the observations of each child's problem solving attempts could be explained by the meta-model without excessive contradictions or glaring omissions.

The analysis of the response map data indicated that while there were important qualitative differences in individual response maps, both within and among children, there were some fundamental similarities in the ways in which some children approached problem solving. Essentially, the patterns observed in different children's responses to the same problem and in each child's response to a range of problems could be described by the nature and extent to which the child was observed to be actively engaged in monitoring his or her cognition, that is, in terms of the four approaches identified in the meta-model. For example, some children were more likely to monitor their cognitive actions than their cognitive goals (Player's). Others were more likely to monitor their cognitive goals and their cognitive and metacognitive knowledge than their cognitive actions (Diver's). An example of a response judged to be characteristic of a Survivor's Approach, that is, *low conceptual/low procedural*, is provided below for the *Incy Wincy* problem referred to earlier. The response map is supported by an abbreviated transcript of interview (Kellie's comments are in bold).

Kellie. . (problem presented orally). **.7 days** ... How did you get that? ... **half of 14 and I thought that might be the answer because if you count by twos to 14 ... you'll get 7** ... Do you remember the problem? ... **Yes** ... Where did the 2 come from? ... **Oh, it was 2 she slid down** ... Do you still think it is 7 days? You'd be fairly happy about that? ... Is there any other way you could do it? **... You could go by 3's to 14, but then you'll have to say when you get to 12 ...if you add another 3 that would be 15, so I'd have to do an extra** ... 3 from? **... oh, here ... from 5 down to ... there was 3 left.** ... How many days will it take Incy Wincy to get to the top of the pipe? **.... 4 and a half days.** ... How did you do that? With the threes was it? **... yes** . ... so how did you get 4? **... well, there are 4 threes in 12, and when I said "and a half", that was including, that meant that he had 2 more jumps to go** *(note link to Freddie context, "jumping")* . ... Would you use a diagram? **.... I might ...** but in your head is the first way you would do it? ... **yes, and then, to check it I might do a picture** ... Do you have a picture in your head like you told me you had for Freddie? **... yes** ... What did you have ... imagine in your head? **... well ... I was thinking every night he went back 2 ... 2 of the steps went away** ... So how many steps did he go up each day? ... **4 each day ... but in 24 hours ... 2 steps** ... That was for Freddie was it? ... **yes** ... What about Incy Wincy? ... **there were 14 steps and then going up 5 and sliding back down 2 and that meant ... it took him one day, then up 5 and down 2 that took him 2 days, and he kept on going like that until he got to the top** ... So, how many did he do in each day? ... **Altogether 3.**

Incy Wincy:			
14 metre pipe	1		
up 5 metres/day	1		
down 2 metres/night	1	O halving, use	
How many days to reach the top?	1	number facts, "7 days"	
2 metres/day (F.F.?)	O	O recalls problem	
Where did the 2 come from?	1	conditions, "oh, it was 2 she slid	
Any other way?	1	down"	O subtract, 3 metres/day, count
metres per day	O		by 3's to 14, "4 and
threes in 14	O		a half days"
Response Map for:	**Kellie**	Approach:	**LC/LP - Survivor**

Figure 3

DISCUSSION

Quite clearly, the link between the conceptual and procedural knowledge is a dynamic and complex one affected by contextual setting, specific content knowledge, beliefs, motivations, and values. The orthogonal relationship proposed in the model was useful in describing and explaining the children's problem solving behaviour. This feature of the model highlighted the fact that many problem solving efforts failed not necessarily because of a lack of monitoring ability *per se* or even a lack of knowledge as to what strategies to use and when, but because of a lack of access to specific content, and the inability to apply monitoring strategies to what is known as opposed to what is done. For example, many children knew that a diagram was useful for the *Incy Wincy* problem but failed to implement this strategy, not because of their inability to draw, but because they did not have access to a well structured knowledge base which could generate and withstand the negotiation required to establish *what* diagram was needed. This confirms an almost identical result reported by Resnick and Nelson-Le Gall (1987) and Lesh's (1983) observation that "the activities that facilitated successful solutions were those which focussed on conceptual rather than procedural considerations" (p.6). As an exploratory study, the work reported here is believed to have important theoretical and methodological implications for future research and classroom practice.

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